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SOME NEW RESULTS RELATED TO THE GENERALIZED SPECIAL FUNCTION OF FRACTIONAL CALCULUS

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ABSTRACT

In the present paper, we introduce two functions namely Ω(c,¬- v,p,q,z)and Ω(c, v,p,q,z) in terms of Advanced M-Series [9] introduced recently by Sharma and show their properties by using fractional integrals and derivatives. Results derived in this paper are the extensions of the results derived earlier by Sharma and Dhakad [7].

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INTRODUCTION

The **Advanced** M-series [9] with $p+2$ upper parameters $a_{1,}a_{2,}\dots a_p, \gamma, \mu$and $q+1$ lower parameters $b_{1,}b_{2,}\dots b_q, \delta$ is

$$ {}_p\overset{\alpha,\beta}{M}_q\left(a_1 \,...\, a_p, \gamma, \mu;, b_1 \,...\, b_q,; z\right) = {}_p\overset{\alpha,\beta}{M}_q(z) $$

$$ {}_p\overset{\alpha,\beta}{M}_q(z) = \sum_{k=o}^{\infty} \frac{(a_1)_k \,...\, (a_p)_k (\gamma)_k (\mu)_k}{(b_1)_k \,...\, (b_q)_k (\delta)_k (n_k)!\,(k)!} \frac{z^k}{\Gamma(\alpha k+\beta)} \qquad (1.1) $$

Here,$\alpha, \beta \epsilon C$, $R(\alpha)>0, m>0$ and $(a_j)_k$ $(b_j)_k$ $,(\gamma)_k(\mu)_k, (\delta)_k$are pochammer symbols. $(n_k)>0$The series (1.1) is defined when none of the denominator parameters $b_j s$, $j=1,2,\dots q$ is a negative integer or zero. If any parameter a_j is negative then the series (1.1))terminates into a polynomial in z. By using ratio test, it is evident that the series (1.1) is convergent for all z, when $q \geq p$, it is convergent for $|z|<1$ when $p=q+1$, divergent when $p>q+1$. In some cases the series is convergent for $z=1, z=-1$. Let us consider take,

$$ \beta = \sum_{j=1}^{p} a_j - \sum_{j=1}^{q} b_j $$

when $p=q+1$, the series is absolutely convergent for $|z|=1$ if $R(\beta)<0$, convergent for $z=-1$, if $0 \leq R(\beta) < 1$ and divergent for $|z|=1$, if $1 \leq R(\beta)$.

Some Special Cases

A) If we put $(\delta)_k = (\mu)_k, n_k = 1$ in equation (1.2)it convertes in k Function[7]

$$ {}_pk_q^{\alpha,\beta,\gamma}\left(a_1 \,...\, a_p;, b_1 \,...\, b_q; z\right) = \sum_{k=o}^{\infty} \frac{(a_1)_k \,...\, (a_p)_k}{(b_1)_k \,...\, (b_q)_k} \frac{(\gamma)_k z^k}{(k)!\,\Gamma(\alpha k+\beta)} \quad (1.2) $$

B) If we put $(\delta)_k = (\mu)_k, n_k = 1, \gamma = 1$ in equation (1.2)it convertes in, Generalized M-Series [10]

$$ {}_{p}\overset{\alpha,\beta}{M}{}_{q}(z) = \sum_{k=o}^{\infty} \frac{(a_1)_k \ ...(a_p)_k}{(b_1)_k \ ...(b_q)_k} \frac{z^k}{\Gamma(\alpha k + \beta)} \quad (1.3) $$

C) If we put $(\delta)_k = (\mu)_k, n_k = 1, \gamma = 1, \beta = 1$ in equation (1.2)it convertes in, M-Series [8]

$$ {}_{p}\overset{\alpha}{M}{}_{q}(z) = \sum_{k=o}^{\infty} \frac{(a_1)_k \ ...(a_p)_k}{(b_1)_k \ ...(b_q)_k} \frac{z^k}{\Gamma(\alpha k + 1)} \quad (1.4) $$

$D)$ ${}_{0}\overset{\alpha,\beta}{M}{}_{0}$ i.e. no p upper or q lower parameters and $(\delta)_k = (\mu)_k, n_k = 1$

$$ {}_{0}\overset{\alpha,\beta}{M}{}_{0}(...;...;z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k (z)^k}{\Gamma(\alpha k + \beta)(k)!} \quad (1.5) $$

Thus the series reduced to the Mittag-Leffler function as in [6].

MATHEMATICAL PREREQSITIES

The Riemann-Liouville fractional integral of order $v \in C$ is defined by Miller and Ross[3] (1993, p.45)

$$ {}_{0}D_t^{-v} f(t) = \frac{1}{\Gamma(v)} \int_0^t (t-u)^{v-1} f(u)du, \quad (2.1) $$

where Re(v)>0. Following Samko et al. [6](1993, p. 37) we define the fractional derivative for $\alpha > 0$ in the form

$$ {}_{0}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{f(u)du}{(t-u)^{\alpha-n+1}}, \qquad (n = [\mathrm{Re}(\alpha)] + 1), \quad \mathbf{(2.2)} $$

Where $[\mathrm{Re}(\alpha)]$ means the integral part of $\mathrm{Re}(\alpha).$

Fractional Calculus Operators and **Advanced M-series**:

Let $f(t) = \sum_{k=o}^{\infty} \frac{(a_1)_k \ ...(a_p)_k (\gamma)_k (\mu)_k}{(b_1)_k \ ...(b_q)_k (\delta)_k (n_k)!(k)!}$

Fractional Integral and Fractional Derivative of the Sharma's Advanced M-series [9]:

Let us consider the fractional Riemann-Liouville (R-L) integral operator (for lower limit $a = 0$ with respect to variable z) of the Advanced M-Series (1.1).

$$ I_z^v f(t) = \frac{1}{\Gamma(v)} \int_0^z (z-t)^{v-1} f(t)\, dt $$

$$= \frac{1}{\Gamma(v)} \int_0^z (z-t)^{v-1} \sum_{k=o}^{\infty} \frac{(a_1)_k \ ...(a_p)_k (\gamma)_k (\mu)_k}{(b_1)_k \ ...(b_q)_k (\delta)_k (n_k)! (k)! (k)!} (ct)^k \, dt$$

$$= \frac{1}{\Gamma(v)} \sum_{k=o}^{\infty} \frac{(a_1)_k \ ...(a_p)_k (\gamma)_k (\mu)_k (c)^k}{(b_1)_k \ ...(b_q)_k (\delta)_k (n_k)! (k)! (k)!} \int_0^z (z-t)^{v-1} t^k \, dt$$

$$= \frac{1}{\Gamma(v)} \sum_{k=o}^{\infty} \frac{(a_1)_k \ ...(a_p)_k (\gamma)_k (\mu)_k (c)^k}{(b_1)_k \ ...(b_q)_k (\delta)_k (n_k)! (k)! (k)!} z^{k+1+v-1} B(k+1, v)$$

$$= \frac{1}{\Gamma(v)} \sum_{k=o}^{\infty} \frac{(a_1)_k \ ...(a_p)_k (\gamma)_k (\mu)_k (c)^k}{(b_1)_k \ ...(b_q)_k (\delta)_k (n_k)! (k)! (k)!} z^{k+v} \frac{\Gamma(k+1)\Gamma(v)}{\Gamma(k+1+v)}$$

$$= z^v \sum_{k=o}^{\infty} \frac{(a_1)_k \ ...(a_p)_k (\gamma)_k (\mu)_k}{(b_1)_k \ ...(b_q)_k (\delta)_k (n_k)! (k)!} \frac{(cz)^k}{(k)!} \frac{\Gamma(k+1)}{\Gamma(k+1+v)}$$

$$= z^v \sum_{k=o}^{\infty} \frac{(a_1)_k \ ...(a_p)_k (\gamma)_k (\mu)_k (cz)^k}{(b_1)_k \ ...(b_q)_k (\delta)_k (n_k)! \, \Gamma(k+1+v) (k)!}$$

$$= z^v \, {}_p\overset{1,1+v}{M}_q (cz)$$

We define $\Omega(\text{ c}, v, p, q, z) = z^v \, {}_p\overset{1,1+v}{M} \ (cz)$

Analogously, $R - L$ fractional derivative operator of the **Advanced M-series [9]** with respect to z.

$$D_z^v f(z) = \frac{1}{\Gamma(\text{n}-v)} \left(\frac{d}{dz}\right)^n \int_0^z (z-t)^{n-v-1} f(t) \, dt$$

$$= \frac{1}{\Gamma(\text{n}-v)} \left(\frac{d}{dz}\right)^n \int_0^z (z-t)^{n-v-1} \sum_{k=o}^{\infty} \frac{(a_1)_k \ ...(a_p)_k (\gamma)_k (\mu)_k}{(b_1)_k \ ...(b_q)_k (\delta)_k (n_k)! (k)! (k)!} (ct)^k dt$$

$$= \frac{1}{\Gamma(\text{n}-v)} \sum_{k=o}^{\infty} \frac{(a_1)_k \ ...(a_p)_k (\gamma)_k (\mu)_k}{(b_1)_k \ ...(b_q)_k (\delta)_k (n_k)! (k)!} \frac{(c)^k}{(k)!} \left(\frac{d}{dz}\right)^n$$

$$\int_0^z (z-t)^{n-v-1} \ t^k \, dt$$

$$= \frac{1}{\Gamma(\text{n}-v)} \sum_{k=o}^{\infty} \frac{(a_1)_k \ ...(a_p)_k (\gamma)_k (\mu)_k}{(b_1)_k \ ...(b_q)_k (\delta)_k (n_k)! (k)!} \frac{(c)^k}{(k)!} \left(\frac{d}{dz}\right)^n$$

$$z^{k+n-v} B(k+1, n-v)$$

We use the modified Beta-function in above equation, which is defined as:

$$\int_a^b (b-t)^{\beta-1} \ (t-a)^{\alpha-1} \, dt \ = (b-a)^{\alpha+\beta-1} B(\alpha, \beta),$$

for $R(\alpha) > 0, R(\beta) > 0$

Again,

$$D_z^v f(z) = \frac{1}{\Gamma(\mathrm{n}-v)} \sum_{k=o}^{\infty} \frac{(a_1)_k \ ...(a_p)_k (\gamma)_k (\mu)_k}{(b_1)_k \ ...(b_q)_k (\delta)_k (n_k)!\,(k)!} \frac{(c)^k}{(k)!} \left(\frac{d}{dz}\right)^n$$

$$z^{k+n-v} \frac{\Gamma(k+1)\Gamma(\mathrm{n}-v)}{\Gamma(k+1+n-v)} \quad (3.3)$$

Where $k+1 > 0, n-v > 0$

Differentiation n times the term z^{k+n-v} and using again $\Gamma(a+k) = (a)_k \Gamma(a)$, representation(3.3) reduces to

$$= \sum_{k=o}^{\infty} \frac{(a_1)_k \ ...(a_p)_k (\gamma)_k (\mu)_k}{(b_1)_k \ ...(b_q)_k (\delta)_k (n_k)!\,(k)!} \frac{\Gamma(k+n-v+1)(c)^k}{\Gamma(\alpha k+\beta)\Gamma(k-v+1)(k)!} z^{k-v} \frac{\Gamma(k+1)}{\Gamma(k+1+n-v)}$$

$$= z^{-v} \sum_{k=o}^{\infty} \frac{(a_1)_k \ ...(a_p)_k (\gamma)_k (\mu)_k}{(b_1)_k \ ...(b_q)_k (\delta)_k (n_k)!\,(k)!\,\Gamma(k-v+1)} (cz)^k$$

$$= z^{-v} \sum_{k=o}^{\infty} \frac{(a_1)_k \ ...(a_p)_k (\gamma)_k (\mu)_k}{(b_1)_k \ ...(b_q)_k (\delta)_k (n_k)!\,(k)!} \frac{(cz)^k}{\Gamma(k-v+1)}$$

$$D_z^v \, {}_p\overset{\alpha,\beta}{M}_q (z) = z^{-v} \, {}_p\overset{1,1\text{-}v}{M}_q (cz) \qquad (3.4)$$

We define $\Omega(\mathrm{c}, -v, p, q, z) = z^{-v} \, {}_p\overset{1,1\text{-}v}{M}_q (cz)$ (3.5)

PROPERTIES OF THE FUNCTIONS $\Omega(\mathrm{C}, v, p, q, z)$ AND $\Omega(\mathrm{C}, -v, p, q, z)$

Theorem 4.1 If c is an arbitrary constant then

$$I_z^\sigma \Omega(\mathrm{c}, v, p, q, z) = \Omega(\mathrm{c}, \sigma + v, p, q, z)$$

Proof:

By the definition of the fractional integral (2.1) we have

$$I_z^\sigma \Omega(\mathrm{c}, v, p, q, z) = \frac{1}{\Gamma(\sigma)} \int_0^z (z-t)^{\sigma-1} \, \Omega(\mathrm{c}, v, p, q, t) \, dt$$

$$= \frac{1}{\Gamma(\sigma)} \int_0^z (z-t)^{\sigma-1} t^v \sum_{k=o}^{\infty} \frac{(a_1)_k \ ...(a_p)_k (\gamma)_k (\mu)_k (ct)^k}{(b_1)_k \ ...(b_q)_k (\delta)_k (n_k)!\,\Gamma(k+1+v)(k)!} \, dt$$

$$= \frac{1}{\Gamma(\sigma)} \sum_{k=o}^{\infty} \frac{(a_1)_k \ ...(a_p)_k (\gamma)_k (\mu)_k}{(b_1)_k \ ...(b_q)_k (\delta)_k (n_k)!\,\Gamma(k+1+v)(k)!} \int_0^z (z-t)^{\sigma-1} t^v (ct)^k \, dt$$

On Simplification and using Beta-function in above equation, we get the desired result

$$I_z^{\sigma}\Omega(\text{ c}, v, p, q, z) = \Omega(\text{ c}, \sigma + v, p, q, z)$$

Theorem 4.2 If c is an arbitrary constant then

$$D_z^{\sigma}\Omega(\text{ c}, v, p, q, z) = \Omega(\text{ c}, v - \sigma, p, q, z)$$

Proof: By the definition of the fractional derivative (2.2), we get

$$D_z^{\sigma}\Omega(\text{ c}, v, p, q, z) = \Omega(\text{ c}, v - \sigma, p, q, z)$$

$$D_z^{\sigma}\Omega(\text{ c}, v, p, q, z) = \frac{1}{\Gamma(\text{n}-\sigma)}\left(\frac{d}{dz}\right)^n \int_0^z (z-t)^{n-\sigma-1}\, t^{-v} \sum_{k=o}^{\infty} \frac{(a_1)_k \ldots (a_p)_k (\gamma)_k(\mu)_k}{(b_1)_k \ldots (b_q)_k (\delta)_k (n_k)!(k)!} \frac{(ct)^k}{\Gamma(k-v+1)} dt$$

$$\Omega(\text{ c}, v, p, q, z) = \frac{1}{\Gamma(\text{n}-\sigma)} \sum_{k=o}^{\infty} \frac{(a_1)_k \ldots (a_p)_k (\gamma)_k(\mu)_k (c)^k}{(b_1)_k \ldots (b_q)_k \Gamma(k-v+1)(\delta)_k (n_k)!(k)!} \left(\frac{d}{dz}\right)^n \int_0^z (z-t)^{n-\sigma-1}\, t^{-v}\, (t)^k dt$$

Simplification and using Beta-function in above equation, we get the desired result

$$D_z^{\sigma}\Omega(\text{ c}, v, p, q, z) = \Omega(\text{ c}, v - \sigma, p, q, z)$$

This completes the analysis.

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